

Quaternions in Three Dimensions

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Abstract. We construct the quaternion algebra [10] “geometrically” by a three dimensional analogue of the classic two dimensional geometric description of the complex field. The algebraic description of the multiplication operation in three dimensions involves the addition of one term. The construction leads to novel methods for implementing and interpolating rotations and understanding their topology.

The bilinear map from ordered pairs of vectors in \mathbb{R}^3 to quaternions, $\mathbf{T} : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{H}$ defined by:

$$\mathbf{T}(\mathbf{v}, \mathbf{w}) = [\mathbf{v} \cdot \mathbf{w}, \mathbf{v} \times \mathbf{w}] \quad (1)$$

is known to be surjective. (For example, see [3], Theorem 1.) Note that in [3] quaternions are viewed as 2×2 , self-adjoint, complex matrices, while ‘vectors’ are identified with elements of the subspace of such matrices having trace zero. However here will view vectors \mathbf{v}, \mathbf{w} as elements of \mathbb{R}^3 and, for the moment, we will pretend that the quaternion structure had not been invented and simply view \mathbf{T} as a map to $\mathbf{Q} = [q, \mathbf{q}] \in \mathbb{R} \times \mathbb{R}^3 \cong \mathbb{R}^4$. Although \mathbf{T} is not injective, if we define an equivalence relation,

$$(\mathbf{v}, \mathbf{w}) \sim (\mathbf{v}', \mathbf{w}') \Leftrightarrow \mathbf{v} \cdot \mathbf{w} = \mathbf{v}' \cdot \mathbf{w}' \text{ and } \mathbf{v} \times \mathbf{w} = \mathbf{v}' \times \mathbf{w}', \quad (2)$$

then we get a bijective map $\tilde{\mathbf{T}}(\mathbf{v}, \mathbf{w}) = \mathbf{T}(\mathbf{v}, \mathbf{w})$ defined on the set $\mathbb{G} = (\mathbb{R}^3 \times \mathbb{R}^3) / \sim$ of its equivalence classes, the inverse images $\mathbf{T}^{-1}[q, \mathbf{q}]$ of \mathbf{T} . It then becomes natural to ask:

“Can we introduce a ‘multiplication’ operation for ordered pairs of vectors that makes the non-zero classes, \mathbb{G}^* , a group, and in such a way that $\tilde{\mathbf{T}}$ is a multiplicative homomorphism with some corresponding operation on $\mathbb{R}^4 - \mathbf{0}$?

Our goal in this note is to show that this question **does** have an affirmative answer, and moreover we will define the multiplication operation by generalizing a very familiar geometric construction. When we are done, we will see that its expression as an operation on $\mathbb{R} \times \mathbb{R}^3$ is $[q', \mathbf{q}'] [q, \mathbf{q}] \rightarrow [q'', \mathbf{q}'']$, where

$$q'' = qq' - \mathbf{q} \cdot \mathbf{q}' \text{ and } \mathbf{q}'' = q\mathbf{q}' + q'\mathbf{q} + \mathbf{q}' \times \mathbf{q}. \quad (3)$$

This formula is a modern representation of the quaternion multiplication formula that first appears in [13]. It is not hard to check that with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denoting the standard basis for \mathbb{R}^3 , if we set $\mathbf{1} = [1, \mathbf{0}], i = [0, \mathbf{e}_1], j = [0, \mathbf{e}_2], k = [0, \mathbf{e}_3]$, then (3) implies the defining relations for quaternions, famously written on a bridge by Hamilton in 1844 [10],

$$i^2 = j^2 = k^2 = -\mathbf{1} \text{ and } ijk = -\mathbf{1}$$

[3]. It follows that $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. Thus our construction makes $\tilde{\mathbf{T}}$ an isomorphism of \mathbb{G}^* and \mathbb{H}^* and gives a purely three-dimensional way to interpret (and even ‘invent’) quaternions. After developing these ideas, we examine in this light the implications of viewing known constructions that relate quaternions and rotations.

The familiar geometric construction to which we refer combines equivalence classes of ordered pairs of points in Euclidean space to define vector addition, and combines equivalence classes of ordered pairs of vectors in the plane to define complex multiplication. (Both of these are treated in detail in [1] where they are referred to as ‘Chasles’ Relation’—though [16] questions this terminology. Similar constructions are also used to construct the ring of integers from the positive integers and the rationals from the integers.) What we will show is that a straightforward generalization of those well-known geometric constructions does indeed make \mathbf{T} a multiplicative isomorphism.

Using $\|\mathbf{v}\|^2 \|\mathbf{w}\|^2 = |\mathbf{v} \cdot \mathbf{w}|^2 + \|\mathbf{v} \times \mathbf{w}\|^2$, if we define $\|\mathbf{Q}\|^2 = q^2 + \|\mathbf{q}\|^2$, we see that the equivalence class $\mathbf{T}^{-1}[q, \mathbf{q}]$ consists of pairs of vectors orthogonal to \mathbf{q} that are separated by a fixed oriented angle $\arccos(\frac{q}{\|\mathbf{Q}\|})$ and the product of whose lengths is $\|\mathbf{Q}\|$. (If $\mathbf{q} = \mathbf{0}$, this says \mathbf{v} and \mathbf{w} are parallel in an arbitrary direction.) An important fact is that any two of these equivalence classes ‘overlap’, by which we mean that there exists a unit vector \mathbf{u} for which there are representatives of both classes having \mathbf{u} as their first element and also a unit vector \mathbf{v} for which there are representatives having \mathbf{v} as second elements. This makes it possible, in analogy with vector addition and complex multiplication, to define the geometric composition of an ordered pair of our equivalence classes, $((\mathbf{v}, \mathbf{w}), (\mathbf{v}', \mathbf{w}'))$ by finding representatives such that the second element of the representative of the first class is equal to the first element of the representative of the second, and then ‘merging’ them into $(\mathbf{v}'', \mathbf{w}'') = (\mathbf{v}', \mathbf{w}') \circ (\mathbf{v}, \mathbf{w})$, the class of the first element of the first and the second of the second. Of course we must show that this geometric merging operation \circ , is well-defined and that it makes the non-zero classes a group isomorphic to the multiplicative group of non-zero quaternions. Both of these facts are a consequence of the following theorem.

Theorem 1. *Let $\mathbf{v}, \mathbf{w}, \mathbf{v}', \mathbf{w}'$ be vectors in \mathbb{R}^3 with $\mathbf{w} = \mathbf{v}'$ and $\|\mathbf{w}\| = 1$. Then*

$$\mathbf{v} \cdot \mathbf{w}' = (\mathbf{v} \cdot \mathbf{w})(\mathbf{v}' \cdot \mathbf{w}') - (\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{v}' \times \mathbf{w}'), \quad (4a)$$

and

$$\mathbf{v} \times \mathbf{w}' = (\mathbf{v} \cdot \mathbf{w})(\mathbf{v}' \times \mathbf{w}') + (\mathbf{v}' \cdot \mathbf{w}')(\mathbf{v} \times \mathbf{w}) + (\mathbf{v}' \times \mathbf{w}') \times (\mathbf{v} \times \mathbf{w}). \quad (4b)$$

Remarks: Before giving the proof, we note that if we put $\mathbf{T}(\mathbf{v}, \mathbf{w}) = [q, \mathbf{q}]$, $\mathbf{T}(\mathbf{v}', \mathbf{w}') = [q', \mathbf{q}']$, and $\mathbf{T}(\mathbf{v}, \mathbf{w}') = [q'', \mathbf{q}'']$ the conclusion becomes equation (3). Also note that the restriction $\|\mathbf{w}\| = 1$ may be removed by an appropriate scaling that puts (4a, b) in a form that highlights their interpretation as a novel vector identity involving just three vectors:

Corollary. For any three vectors $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^3$,

$$(\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})(\mathbf{B} \cdot \mathbf{C}) - (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}), \quad (4a')$$

$$(\mathbf{A} \times \mathbf{C})(\mathbf{B} \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})(\mathbf{B} \times \mathbf{C}) + (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \times \mathbf{B}) + ((\mathbf{B} \times \mathbf{C}) \times (\mathbf{A} \times \mathbf{B})). \quad (4b')$$

While there appears to be an asymmetry, with \mathbf{B} plays a distinguished role, \mathbf{A}, \mathbf{B} , and \mathbf{C} can be cyclically permuted. We will see that these remarkable vector identities contain

the composition laws for (i)the quaternions, \mathbb{H} , (ii)the unit quaternions, $SU(2)$, (iii)the complex numbers, \mathbf{C} , (iv)the plane rotation group $SO(2)$, and (v)the three-dimensional rotation group, $SO(3)$!

Proof: The identities can be validated by direct brute force comparison of the two sides, but since our purpose is to derive them, we prefer a geometric approach. By the orientation preserving Givens' **QR** factorization algorithm for the matrix with columns $\{\mathbf{v}, \mathbf{w}, \mathbf{w}'\}$, we can always construct a positively oriented orthonormal ordered basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for \mathbb{R}^3 such that $\mathbf{e}_1 = \mathbf{w}$ and \mathbf{v} is in the $\mathbf{e}_1 - \mathbf{e}_2$ -plane. To motivate (4a,b) we first assume \mathbf{w}' is co-planar with \mathbf{v} and \mathbf{w} , i.e., also in the $\mathbf{e}_1 - \mathbf{e}_2$ -plane. In coordinates with respect to this basis, $\mathbf{q} = \langle 0, 0, \|\mathbf{q}\| \rangle$, $\mathbf{q}' = \langle 0, 0, \|\mathbf{q}'\| \rangle$ so $\mathbf{v} = \langle q, -\|\mathbf{q}\|, 0 \rangle$, $\mathbf{w} = \langle 1, 0, 0 \rangle$ and $\mathbf{w}' = \langle q', \|\mathbf{q}'\|, 0 \rangle$. We confirm by direct calculation that

$$q'' = \mathbf{v} \cdot \mathbf{w}' = qq' - \mathbf{q} \cdot \mathbf{q}', \quad \mathbf{q}'' = \mathbf{v} \times \mathbf{w}' = q\mathbf{q}' + q'\mathbf{q} \quad (5)$$

which we recognize from the geometric meaning of dot and cross products as complex multiplication: $[\cos(\theta)\cos(\theta') - \sin(\theta)\sin(\theta'), \langle 0, 0, \cos(\theta)\sin(\theta') + \cos(\theta')\sin(\theta) \rangle]$. We now let \mathbf{w}' be arbitrary. Then, since \mathbf{q}' remains orthogonal to $\mathbf{w} = \mathbf{e}_1$, it takes the form $\mathbf{q}' = \langle 0, \|\mathbf{q}'\|\cos(\phi), \|\mathbf{q}'\|\sin(\phi) \rangle$ and so $\mathbf{w}' = \langle q', \|\mathbf{q}'\|\sin(\phi), -\|\mathbf{q}'\|\cos(\phi) \rangle$. Again by direct computation,

$$q'' = \mathbf{v} \cdot \mathbf{w}' = qq' - \mathbf{q} \cdot \mathbf{q}', \quad \mathbf{q}'' = \mathbf{v} \times \mathbf{w}' = q\mathbf{q}' + q'\mathbf{q} + \langle \|\mathbf{q}\| \|\mathbf{q}'\| \cos(\phi), 0, 0 \rangle. \quad (6)$$

To complete the derivation and proof of (4a,b) we observe that the given data must determine this cross-product in terms of equivariant quantities and recognize the last term of (5) as the non-commutative quantity $\mathbf{q}' \times \mathbf{q}$. Since in (4a,b) the dot-product is invariant ($\mathbf{R}\mathbf{x} \cdot \mathbf{R}\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$) and the cross-product equivariant ($\mathbf{R}\mathbf{x} \times \mathbf{R}\mathbf{y} = \mathbf{R}(\mathbf{x} \times \mathbf{y})$), they remain valid not only when the basis or triple of vectors are simultaneously rotated, but also under any rotation of the pairs (\mathbf{v}, \mathbf{w}) and $(\mathbf{v}', \mathbf{w}')$ within their respective equivalence classes. This completes the derivation and proof of (4a,b), and guarantees that our geometric merging operation \circ is well defined on \mathbb{G} , and that with respect to this operation, $\tilde{\mathbf{T}}$ is a multiplicative isomorphism. **QED.**

What we are doing here is viewing quaternion multiplication as the expression of the relationships (4a,b) that hold for the dot-product and cross-products of each pair of a triple of vectors in \mathbb{R}^3 . (Surprisingly, (4a,b) do not appear in standard collections of similar vector identities.) While this seems to be novel, if we restrict the vectors in \mathbb{G} to lie in any two-dimensional subspace, the final term in (4b), (4b') vanishes and $\tilde{\mathbf{T}}$ becomes equivalent to Wessel's construction of an isomorphism between the nonzero elements of $(\mathbb{R}^2 \times \mathbb{R}^2)/\sim$ and the multiplicative group of complex numbers, $\mathbf{C} - \{0\}$. We obtain a novel description of the three-sphere by restricting the vectors in \mathbb{G} to be unit vectors, in which case the $(\mathbf{B} \cdot \mathbf{B})$ factors in (4a'), (4b') vanish, and $\tilde{\mathbf{T}}$ becomes an isomorphism between $(S^2 \times S^2)/\sim$ and the unit quaternions, $SU(2) \cong S^3$. In this case we may also associate ordered pairs of unit vectors with oriented arcs of great circles in the unit sphere $S^2 \subset \mathbb{R}^3$, in which case multiplication of unit quaternions may be viewed as 'vector addition' of these arcs. The unit quaternion inverse to (\mathbf{v}, \mathbf{w}) corresponds to (\mathbf{w}, \mathbf{v}) and the oppositely oriented

arc. Algebraically, the dot-product stays the same while the cross-product is multiplied by -1 . Under the combination of the above restrictions, the final term in (4b), (4b') and the $(\mathbf{B} \cdot \mathbf{B})$ factors in (4a'), (4b') vanish, in which case they revert to the standard trigonometric addition formulas, and $\tilde{\mathbf{T}}$ becomes an isomorphism between $(S^1 \times S^1)/\sim$ and the unit circle group, $U(1) \cong SO(2)$.

Continuing the analogy with vectors and complex numbers, we note that the additive structure of quaternions is also captured geometrically by \mathbb{G} , using the fact noted earlier that any two equivalence classes overlap with representatives of the corresponding equivalence classes that have the same first elements: $\mathbf{Q} = \mathbf{T}(\mathbf{v}, \mathbf{w}), \mathbf{Q}' = \mathbf{T}(\mathbf{v}, \mathbf{w}')$. Linear combinations of the second element are preserved by \mathbf{T} because of the bilinearity of both dot-products and cross-products, i.e., $c\mathbf{Q} + c'\mathbf{Q}' = \mathbf{T}(\mathbf{v}, c\mathbf{w} + c'\mathbf{w}')$. In particular a convex combination (and so interpolation) is preserved. But even more is true ! We can show that ‘SLERP’ (quaternion) interpolation [14] on S^3 :

$$\mathbf{Q}(t) = c(t)\mathbf{Q} + c'(t)\mathbf{Q}', \quad c(t) = \frac{\sin((1-t)\Omega)}{\sin(\Omega)}, \quad c'(t) = \frac{\sin(t\Omega)}{\sin(\Omega)}, \quad \Omega = \arccos(\mathbf{Q} \cdot \mathbf{Q}') \quad (7)$$

can be performed entirely on S^2 . To see this, note that $\mathbf{Q}(t) = \mathbf{T}(\mathbf{v}, c(t)\mathbf{w} + c'(t)\mathbf{w}')$ with the same $c(t), c'(t)$ as in (7). Then since

$$\mathbf{Q} \cdot \mathbf{Q}' = qq' + \mathbf{q} \cdot \mathbf{q}' = (\mathbf{v} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{w}') + (\mathbf{v} \times \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w}'),$$

the Lagrange-Binet-Cauchy identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \quad (8)$$

reduces this to $\mathbf{Q} \cdot \mathbf{Q}' = \mathbf{w} \cdot \mathbf{w}'$. In other words $c(t)$ and $c'(t)$ do not change if we replace the definition of Ω in (7) with $\Omega = \arccos(\mathbf{w} \cdot \mathbf{w}')$.

This viewpoint also provides insight into the well-known double-covering homomorphism of $SU(2)$, the group of unit quaternions, onto the group $SO(3)$ of rotations of \mathbb{R}^3 . This takes $\mathbf{Q} \in SU(2)$ to $\mathbf{R}(\mathbf{Q}) \in SO(3)$ defined by conjugation: $[c, \mathbf{v}] \mapsto \mathbf{Q}[c, \mathbf{v}]\mathbf{Q}^{-1} = [c, \mathbf{R}(\mathbf{Q})\mathbf{v}]$, or equivalently by the Euler-Rodrigues-Shoemake formula [9]:

$$\mathbf{R}(\mathbf{Q}) = \mathbf{I} + 2q\mathbf{J}_{\mathbf{q}} + 2\mathbf{J}_{\mathbf{q}}^2, \quad \text{where } \mathbf{J}_{\mathbf{u}} := \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}. \quad (9)$$

When we view a unit quaternion as $\overline{(\mathbf{v}, \mathbf{w})} \in \mathbb{G}$, this homomorphism take the simple form $\tilde{C} : \overline{(\mathbf{v}, \mathbf{w})} \mapsto C(\mathbf{v}, \mathbf{w}) = \rho_{\mathbf{w}}\rho_{\mathbf{v}}$, where $\rho_{\mathbf{u}} = 2\mathbf{u}\mathbf{u}^T - \mathbf{I}$ is the orthogonal reflection across the one-dimensional subspace spanned by the unit vector \mathbf{u} . The fact that the double covering map \tilde{C} is well-defined on equivalence classes of (2), is a consequence of the relationship between rotations \mathbf{R} that fix $\mathbf{q} = \mathbf{v} \times \mathbf{w}$, and reflections $\rho_{\mathbf{u}}$ with $\mathbf{u} \cdot \mathbf{q} = 0$:

$$\rho_{\mathbf{Ru}} = \mathbf{R}\rho_{\mathbf{u}}\mathbf{R}^{-1}. \quad (10)$$

Then the fact that \tilde{C} is a homomorphism is a consequence of the idempotency of the reflections in the unit vector, \mathbf{u} , used to compose two classes, $(\rho_{\mathbf{u}})^2 = \mathbf{I}$. Because $\rho_{-\mathbf{u}} = \rho_{\mathbf{u}}$, $C(\mathbf{v}, \mathbf{w}) = C(-\mathbf{v}, \mathbf{w})$ where $(-\mathbf{v}, \mathbf{w}) \sim (\mathbf{v}, -\mathbf{w})$ both represent the negative of (\mathbf{v}, \mathbf{w}) . As oriented arcs, they are represented by the arc complementary to that of (\mathbf{v}, \mathbf{w}) , from which $2(\theta + \pi) = 2\theta \pmod{2\pi}$ also shows that they correspond to the same rotation. The remaining fact, that its kernel consists **only** of the two classes (\mathbf{u}, \mathbf{u}) and $(\mathbf{u}, -\mathbf{u})$ for any $\mathbf{u} \in S^2$ follows from the observation that $\rho_{\mathbf{u}}\mathbf{u} = \mathbf{u}$ and $\rho_{\mathbf{v}}\mathbf{v} = \mathbf{v}$ and therefore if \mathbf{v} is not a multiple of \mathbf{u} , $C(\mathbf{u}, \mathbf{v})$ cannot fix either \mathbf{u} or \mathbf{v} . Therefore we may view $SO(3)$ as equivalence classes of lines through the origin, with a geometric composition analogous to \circ , and an algebraic composition derived from (4a), (4b) by identifying classes with opposite signs.

The map C is related to two algorithms for implementing the unique rotation \mathbf{R} that takes a unit vector \mathbf{u}_I to another unit vector $\mathbf{u}_F \neq -\mathbf{u}_I$ and fixes the orthogonal complement of their span. The first, motivated by Cartan's 'transvection' isometry [2], has been used for an especially efficient implementation of a virtual trackball [5] in the mathematical visualization projects, 3D-XplorMath [11] and the Virtual Math Museum [12]. It defines $\mathbf{s} = \mathbf{u}_I + \mathbf{u}_F$ and with implicit normalization to avoid square roots, then applies \mathbf{R}^{-1} to the vectors \mathbf{e} of the viewing frame in the form $\rho_{\mathbf{s}}(\rho_{\mathbf{u}_F}\mathbf{e})$. This highlights the fact that the angle between any vector and its image under the rotation $C(\mathbf{v}, \mathbf{w})$ is *twice* the angle between \mathbf{v} and \mathbf{w} . This observation makes it possible to compose two three-dimensional rotations "with our fingers", by composing the corresponding unit quaternions, thus forming a virtual $SO(4)$ (or, depending on interpretation, $SU(2)$) "slide rule". If you represent the two planes and half-angles of the rotations using the thumbs and forefingers of your two hands, and then find the equivalent configuration so that the interior thumb and forefinger overlap, then the exterior thumb and forefinger represent the plane and half-angle of the composed rotation! It also explains the difference between the quarter-turn rotation about \mathbf{e}_3 corresponding to the complex i and the half-turn rotation about \mathbf{e}_1 corresponding to the quaternion i . The other algorithm explicitly calculates the matrix for \mathbf{R} in a novel form:

$$\mathbf{R} = \mathbf{I} - \frac{2}{(\mathbf{s} \cdot \mathbf{s})} \mathbf{s} \mathbf{s}^T + 2\mathbf{u}_F \mathbf{u}_I^T. \quad (11)$$

This can be implemented using only 18 multiplications and one division, and unlike the formula in [7], (11) remains valid in any dimension!

Finally, we observe that the intersection of S^2 with the family of planes in \mathbb{R}^3 containing the standard basis vector \mathbf{e}_1 whose normals vary at unit speed along the geodesic from \mathbf{e}_3 to \mathbf{e}_1 is

$$\mathbf{e}(s, t) = \sin(t/4)(\sin(t/4)\mathbf{e}_1 + \cos(t/4)\mathbf{e}_3) + \cos(t/4)(\cos(s)(-\cos(t/4)\mathbf{e}_1 + \sin(t/4)\mathbf{e}_3) + \sin(s)\mathbf{e}_2).$$

This explicit one-parameter family of closed paths on S^2 (with base-point \mathbf{e}_1) starts from one turn about the equator in the $\mathbf{e}_1 - \mathbf{e}_2$ -plane, namely $\mathbf{e}(s, 0) = \cos(s)\mathbf{e}_1 + \sin(s)\mathbf{e}_2$, and shrinks smoothly to the constant path $\mathbf{e}(s, 2\pi) = \mathbf{e}_1$. Then $\mathbf{R}(s, t) = C(\mathbf{e}_1, \rho_{\mathbf{e}(s, t)})$ and $\mathbf{Q}(s, t) = \mathbf{T}(\mathbf{e}_1, \rho_{\mathbf{e}(s, t)})$ are explicit homotopies of the paths in $SO(4)$ and in S^3 , respectively, between two turns about a fixed axis (\mathbf{e}_3) and the constant path at the

identity. These deformations are the basis of the famous Dirac Belt Trick [15], Feynman Wine Glass Trick, and the orientation entanglement relation demonstration described in [6] and visualized in [8] and [12]. All of these express, in different ways, the fact that the fundamental group of $SO(4)$ is isomorphic to Z_2 , but the current perspective relates them to the less mysterious fact that S^2 is simply connected, and also suggests why the same construction fails for an odd number of turns or if the axis is constrained to be fixed during the deformation, i.e., for plane rotations.

References

- [1] M. Audin, *Geometry*, Springer, New York, 2003.
- [2] E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Gauthier-Villars, Paris, 1928.
- [3] W. Eberlein, The Geometric Theory of Quaternions, *This Monthly*. **9** (1963) 952–954.
- [4] W. R. Hamilton, On a new species of imaginary quantities connected with a theory of quaternions, *Proceedings of the Royal Irish Academy*, **2** (1843) 424–434.
- [5] K. Henriksen, J. Sporning, K. Hornbaek. Virtual Trackballs Revisited, *IEEE Transaction on Visualization and Computer Graphics*, **2** (2004) 206–216.
- [6] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, Freeman, 1973.
- [7] T. Möller and J. F. Hughes, Efficiently Building a Matrix to Rotate One Vector to Another. *Journal of Graphics Tools* **4** (1999) 1–4.
- [8] B. Palais, Understanding Quaternions Geometrically in \mathbb{R}^3 ., *LOCI*, To appear.
The relevant demos may be found at the following URLs:
<http://www.math.utah.edu/%7Epalais/quaternion%5Frepresentatives.html>
<http://www.math.utah.edu/%7Epalais/quaternion%5Fcomposition.html>
<http://www.math.utah.edu/%7Epalais/quaternion%5Finterpolation.html>
- [9] B. Palais and R. Palais, Euler’s fixed point theorem: The axis of a rotation, *J. Fixed Point Theory Appl.* **2** (2007) 215–220.
- [10] R. Palais, Classification of Real Division Algebras, *This Monthly*. **75** (1964) 366–8.
- [11] R. Palais, et. al., <http://3D-XplorMath.org>
- [12] R. Palais, et. al., <http://VirtualMathMuseum.org>
- [13] O. Rodrigues, Des lois géométriques qui régissent les déplacements d’un système solide dans l’espace, *J. Mathématique Pures et Appliquées*. **5** (1840) 380–440.
- [14] K. Shoemake, Animating Rotation with Quaternion Curves, *Computer Graphics*, **19**, (1985) 245–254.
- [15] V. Stojanoska and O. Stoytchev, Touching the Z_2 in Three-Dimensional Rotations, *Math Magazine* **81** **5** (2008) 345–357.
- [16] R. Zenth, Chasles’ Relation., <http://iml.univ-mrs.fr/%7Eeritzenth/divers-loisirs/chasles2.pdf>.